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Real Clifford algebras and their representations over the reals

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Abstract. The real Clifford algebras have been classified for a long time. We recall the classification and use it to describe all their representations over the reals.

1. Introduction

In a famous paper [1], Brauer and Weyl described the representations of the Clifford algebras over the complex field, but there does not seem to be an equivalent reference for the real case. Our purpose is to give an explicit description of the representations of the real Clifford algebras over the reals.

For every pair of positive integers (p, q), we shall denote by $\mathrm{C}(p, q)$ the Clifford algebra of the non-degenerate real bilinear symmetric form of rank $p+q$ and signature $p-q$. Recall that this is a 2^{p+q}-dimensional real algebra with generators $e_1, \ldots, e_{p+q}$ satisfying the following relations:

$$\begin{aligned} &e_i^2 = 1 &&\text{for} \quad i = 1, \ldots, p \\ &e_i^2 = -1 &&\text{for} \quad i = p+1, \ldots, p+q \\ &e_i e_j = -e_j e_i &&\text{for} \quad i \neq j. \end{aligned} \tag{1}$$

2. Classification

There is a well known classification of these algebras, showing a modulo-8 periodicity in their structure.

Let us first recall three isomorphisms [2-4]:

$$\begin{aligned} &\mathrm{C}(p, q) \approx \mathrm{C}(q, p-2) \otimes \mathrm{C}(2, 0) &&\text{for} \quad p \geqslant 2 \\ &\mathrm{C}(p, q) \approx \mathrm{C}(q-2, p) \otimes \mathrm{C}(0, 2) &&\text{for} \quad q \geqslant 2 \\ &\mathrm{C}(p, q) \approx \mathrm{C}(p-1, q-1) \otimes \mathrm{C}(1, 1) &&\text{for} \quad p, q \geqslant 1 \end{aligned}$$

where all the tensor products are over the reals.

Looking for the structure of an arbitrary real Clifford algebra, we can use them to get the following reductions (see [5,6] for more details). If q is greater than or equal to p, writing $q - p = 8s + r$, with r, s integer and $0 \leqslant r < 8$, we have

$$\mathrm{C}(p, q) \approx \mathrm{C}(1, 1)^{\otimes p} \otimes \mathrm{C}(0, 8)^{\otimes s} \otimes \mathrm{C}(0, r). \tag{2}$$

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Table 1.

n	$C(0, n)$	$C(n, 0)$
0	R	R
1	C	$R\oplus R$
2	H	$M_2(R)$
3	$H\oplus H$	$M_2(C)$
4	$M_2(H)$	$M_2(H)$
5	$M_4(C)$	$M_2(H)\oplus M_2(H)$
6	$M_8(R)$	$M_4(H)$
7	$M_8(R)\oplus M_8(R)$	$M_8(C)$
8	$M_{16}(R)$	$M_{16}(R)$

If p is greater than or equal to q, again writing $p-q=8s+r$, with $0\leqslant r<8$, we have this time

$$C(p, q)\approx C(1, 1)^{\otimes q}\otimes C(8, 0)^{\otimes s}\otimes C(r, 0).$$

On the other hand, we collect in table 1 the structure of low-dimensional algebras which can be found by direct calculation [2].

We denote by R, C, H and $M_r(K)$ the algebras of the real numbers, the complex numbers, the quaternions and the $(r\times r)$ matrices over K, respectively. The classification is then completed by saying that $C(1, 1)$ is isomorphic to $M_2(R)$.

3. Representations

Since all these algebras are finite-dimensional semi-simple, their representations are all direct sums of irreducible ones [8]. The number of equivalence classes of irreducible representations of $C(p, q)$ is one if $p-q$ is not congruent to 1 modulo 4 (the algebra is simple in this case) and two if $p-q$ is congruent to 1 modulo 4 (the algebra is the direct sum of two simple algebras).

It is immediate from the above classification to calculate their dimensions: denoting by $d(p, q)$ the dimension of any irreducible representation of $C(p, q)$, we obtain table 2.

The general case is as follows. If q is greater than or equal to p, writing $q-p=8s+r$ with $0\leqslant r<8$, we have $d(p, q)=2^{p+4s}d(0, r)$, and if p is greater than or equal to q, writing $p-q=8s+r$ with $0\leqslant r<8$, we have this time $d(p, q)=2^{q+4s}d(r, 0)$.

Table 2.

n	$d(0, n)$	$d(n, 0)$
0	1	1
1	2	1
2	4	2
3	4	4
4	8	8
5	8	8
6	8	16
7	8	16
8	16	16

We can now give a description of these representations. We were inspired by [7] who give a similar description for the case $q=0$, but without using any matrix. We shall express the image of the generators e_i of the Clifford algebras as a tensor product of real (2×2) matrices σ_0, σ_1, σ_2 and σ_3, satisfying $\sigma_0=1$, $\sigma_1^2=1$, $\sigma_2^2=-1$, $\sigma_3^2=1$ and $\sigma_i\sigma_j=-\sigma_j\sigma_i$ if $0\neq i\neq j\neq 0$. Take, for example,

$$\sigma_0=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \sigma_2=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \qquad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

We get at once the irreducible representation of C(1, 1):

$$\begin{aligned}&\alpha : \mathrm{C}(1,1)\to \mathrm{M}_2(R)\\ &f_1\mapsto \sigma_1\\ &f_2\mapsto\sigma_2\end{aligned} \tag{3}$$

where f_1, f_2 are the two generators of the presentation (1).

Because of the modulo-8 periodicity in the structure of the algebras, the basic representations are those of C(0, 8) and C(8, 0). We write $g_1,\ldots,g_8$ and $h_1,\ldots,h_8$, respectively, their generators of the presentation (1):

$$\begin{aligned}&\beta : \mathrm{C}(0,8)\to \mathrm{M}_{16}(\mathrm{R})\\ &g_1\mapsto \sigma_2\otimes 1\otimes 1\otimes 1\\ &g_2\mapsto \sigma_3\otimes\sigma_2\otimes 1\otimes 1\\ &g_3\mapsto \sigma_1\otimes\sigma_2\otimes\sigma_1\otimes 1\\ &g_4\mapsto \sigma_3\otimes\sigma_3\otimes\sigma_2\otimes 1\\ &g_5\mapsto \sigma_1\otimes 1\otimes\sigma_2\otimes\sigma_1\\ &g_6\mapsto \sigma_3\otimes\sigma_1\otimes\sigma_2\otimes\sigma_1\\ &g_7\mapsto \sigma_1\otimes\sigma_2\otimes\sigma_3\otimes\sigma_1\\ &g_8\mapsto \sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_2\end{aligned} \tag{4}$$

and

$$\begin{aligned}&\tau : \mathrm{C}(8,0)\to \mathrm{M}_{16}(\mathrm{R})\\ &h_1\mapsto \sigma_1\otimes 1\otimes 1\otimes 1\\ &h_2\mapsto \sigma_3\otimes\sigma_1\otimes 1\otimes 1\\ &h_3\mapsto \sigma_3\otimes\sigma_3\otimes\sigma_1\otimes 1\\ &h_4\mapsto \sigma_2\otimes\sigma_1\otimes\sigma_2\otimes 1\\ &h_5\mapsto \sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_1\\ &h_6\mapsto \sigma_2\otimes 1\otimes\sigma_1\otimes\sigma_2\\ &h_7\mapsto \sigma_3\otimes\sigma_2\otimes\sigma_1\otimes\sigma_2\\ &h_8\mapsto \sigma_2\otimes\sigma_1\otimes\sigma_3\otimes\sigma_2.\end{aligned}$$

The representations of C(0, n) and C(n, 0) with $n<8$ can easily be found from those of C(0, 8) and C(8, 0), respectively. One has to consider the first n generators and take the first factors in their images, so as to get representations of dimension $d(0,n)$ and $d(n,0)$, respectively.

From the above representations, we can construct those of an arbitrary Clifford algebra $C(p, q)$. We treat in detail the case $q \geqslant p$. The case $p \geqslant q$ is similar and will be left to the reader. We write $q-p=8s+r$ with $0 \leqslant r < 8$ and as indicated by the decomposition (2) we put together p representations α of $C(1, 1)$, s representations β of $C(0, 8)$ and a representation β' of $C(0, r)$. We do it the following way, using (3) and (4):

$$
\begin{aligned}
&C(p, q) \to M_{d(p,q)}(R) \\
&e_{2l+i} \mapsto \sigma_3^{\otimes l} \otimes \alpha(f_i) && l=0, 1, \ldots, p-1;\ i=1, 2 \\
&e_{2p+8m+j} \mapsto \sigma_3^{\otimes p+4m} \otimes \beta(g_j) && m=0, 1, \ldots, s-1;\ j=1, \ldots, 8 \\
&e_{2p+8s+k} \mapsto \sigma_3^{\otimes p+4s} \otimes \beta'(g_k) && k=1, \ldots, r.
\end{aligned}
$$

Each image should be completed by tensor products of identity matrices to have the correct dimension. It is obvious that this is the desired representation of $C(p, q)$, these images satisfying the relations (1).

As already pointed out, whenever $p-q$ is congruent to 1 modulo 4, there are two equivalence classes of irreducible representations of $C(p, q)$, so we have to give a second representation. We use the fact (directly checked on the above descriptions) that in this case the image of the product $e_1 e_2 \ldots e_{p+q}$ is ± 1. If we change the image of e_{p+q}, multiplying it be a factor -1, we get a second representation, which cannot be equivalent to the first one, because the product $e_1 e_2 \ldots e_{p+q}$ acts trivially in one representation and not in the other.

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